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A THEORY FOR AERODYNAMIC FORCES AND MOMENTS

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I. INTRODUCTION

The problem of predicting aerodynamic forces and moments acting on finite solid bodies immersed in and moving relative to a fluid has occupied the center stage of aerodynamic research from the end of the nineteenth century onwards. In fact, it is this focal problem that distinguishes the science of aerodynamics from other branches of theoretical fluid mechanics. Studies of the motion of the fluid relative to the solid bodies of course represent a fundamental aspect of aerodynamics. In the majority of aerodynamic applications, however, such studies do not represent ends in themselves. Rather, they are undertaken in recognition of the fact that the motion of the fluid is ultimately responsible for the forces and moments exerted on the solid bodies by the fluid.

It is known that the problem of finding analytical solutions to equations of fluid motion associated with solid configurations of practical importance presents considerable, often insurmountable, mathematical difficulties. Historically, therefore, the most remarkable advances in aerodynamics were brought about by aerodynamicists who perceived approaches for the prediction of aerodynamic forces and moments that avoid, as much as possible, entanglement with the details of the fluid motion. For example, the Kutta-Joukowski theorem, i.e. the circulation theory for two-dimensional steady motion, permits the lift force acting on a solid body to be determined from a knowledge of the circulation about the body. For the case of a two-dimensional body with a sharp trailing edge, the Kutta condition, requiring the rear stagnation point of a potential flow to be located at the trailing edge, is known to yield acceptably accurate values of the circulation, provided that the flow is steady and does not separate over an appreciable region around the solid body. With the Kutta condition,

the problem of predicting the lift force for such a flow is reducible to that of solving an integral equation (Ref. 1). The unknown function of this integral equation is a distribution of singularity (sources, sinks, and vortices) over the body surface. There is no need to know the fluid motion away from the body except that it is reasonably represented by a potential flow about the body.

Extensions of the circulation theory to three-dimensional and to unsteady flows are generally based on the concept of bound vortices and free vortices. The bound vortices are considered to move with the solid body. The free vortices are considered to be shed from the solid body either because of the Helmholtz theorem, which states that a vortex filament cannot begin or terminate within the fluid domain, or because the total circulation of the entire fluid system is required to be zero. These free vortices are located at finite distances from the solid bodies and they make a finite contribution to the fluid motion near the solid surfaces. This contribution is quantitatively determinate once the spatial distribution of the free vortices is known. This distribution is dependent on the complex processes of shedding of vortices and subsequent transport of free vortices in the fluid. To avoid the detailed computation necessary for an accurate determination of this distribution, most previous authors have prescribed the motion of free vortices in a simple manner. For example, with Prandtl's lifting-line theory, it is usually assumed that the free vortices remain stationary relative to the freestream.

The circulation theory and its extensions permit reasonably accurate predictions of the lift force for some solid configurations and flow environments. The scope of applicability of this theory and its extensions has not been precisely established. It is well-known, however, that considerable uncertainties exist regarding the value of the circulation to be prescribed

in cases where the solid body does not possess a sharp trailing edge, where appreciable regions of flow separation exist, or where the motion of the solid is time-dependent. These uncertainties arise mainly because assumptions, or hypotheses, utilized in the development of the theory are often ad-hoc and, consequently, the theory is not readily interpreted as an approximation of a specific physical phenomenon. For example, the bound vortex is usually described in well-known treatises as "replacing" an airfoil (or a wing). The vorticity, defined as the curl of the velocity, is twice the angular velocity in a solid region. If the airfoil is not rotating, then clearly it does not possess vorticity. The bound vortex is therefore not an approximation of the airfoil. To the experienced aerodynamicist, the remarkable agreement between the predicted and measured lift forces under certain circumstances points to the inevitable conclusion that the circulation theory is a reasonable approximation of the physics of the problem under these circumstances. It was suggested (Ref. 2) that the bound vortex does represent the airfoil plus its boundary layers. The fact that such an interpretation was emphasized so very recently (in 1976) has motivated the principal investigator to examine critically the historical development of the circulation theory, together with its more recent extensions, with the hope of facilitating the interpretation of the circulation theory and thus contributing to its further development for three-dimensional flows, unsteady flows, and separated flows. In the course of this study, several general formulas relating aerodynamic forces and moments to the time-variation of vorticity-moment integrals were uncovered. It became evident that these formulas have far reaching consequences in the realms both of theoretical aerodynamics and of computational aerodynamics. In particular, it was shown that the formulas form a theory for aerodynamic forces and moments. This theory encompasses much of the existing aerodynamic theories. For example, the circulation theory for

steady lift and its various extensions are readily interpreted as various levels of approximation of the general theory. The purpose of this report is to present this general theory.

A distinguishing feature of the present theory is that the concept of bound vortex, or that of singularity elements such as sources, sinks, and vortex filaments, is not embodied in the general formulas forming the theory. Rather, the actual vorticity distributions of the fluid and of the solid, the latter being related to the rotational motion of the solid, enter these formulas. The starting point of the present theory is a rotational flow analysis. Consequently, the theory is applicable to viscous flows. This freedom from "bondage" is important in the interpretation of the various aerodynamic theories. For example, it permits a precise definition of the "circulation" about a two-dimensional solid in the case where an appreciable region of separation exists. While the Kutta-Joukowski theorem predicts zero drag, the general formula does relate the time-variation of a vorticity-moment integral to a non-zero drag, including the profile drag. The formulas predict an unsteady drag without the customary energy or apparent mass consideration. These general formulas, in fact, clearly point out the basis principles for minimizing the drag and for maximizing the lift. Many of the measures proposed for drag-reduction and for lift-augmentation are readily interpretable on the basis of these principles. The present research deals only with incompressible flows, although the basic principles described here are certainly applicable to compressible flows as well.

In recent years, extensive efforts have been in progress at many research institutions to develop numerical methods for the solution of aerodynamic problems. An ultimate goal of these efforts is to make available methods of predicting aerodynamic forces and moments that are more accurate and that possess a wider scope of validity than the circulation theory. These efforts are divisible into two major categories. In one category, numerical methods are being developed

for the solution of inviscid flow equations. The free vortices are assumed to convect, but not to diffuse, with the fluid. The solid body is represented by a singularity distribution. Conceptually, these numerical methods utilize the basic assumptions of the circulation theory. They relax the restrictions of analytical methods based on the circulation theory through detailed computation. For example, the linearization procedure introduced in classical studies of the unsteady two-dimensional airfoil problem (e.g. Ref. 3) is no longer necessary if numerical methods (e.g. Ref. 4) are employed. These numerical methods are of course expected to be subject to the well-known limitations of the inviscid flow assumption. In this regard, the availability of the general formulas are expected to offer clearer interpretation of these numerical methods and better definition of their scope of application.

In the second category, numerical methods are being developed for the solution of differential equations governing viscous flows. Impressive progress has been made in recent years in the numerical solution of two-dimensional laminar separated flow problems as well as in the establishment of turbulence models for separated flows. For three-dimensional separated flows, the development of numerical methods is hindered by excessive computation requirements (Ref. 5). Methods that possess superior computational efficiency are therefore of critical importance. During the past few years, the principal investigator and his co-workers have developed a new numerical approach which permits the confinement of the solution field to the vortical region of the flow (Ref. 6). In comparison to available finite-difference and finite-element methods, all of which require the solution field to include the potential region in addition to the vortical region, the new approach requires a drastically smaller number of grid points. The computation requirements using this new approach, called the integro-differential approach, are consequently drastically smaller than

those using other methods (Ref. 7,8). The new approach uses the vorticity vector as a field variable in place of the pressure. The general formulas presented in this report provide a convenient means of computing the aerodynamic forces and moments directly from vorticity distributions. That is, using these general formulas, it is no longer necessary to compute first the pressure and shear stress distributions on the solid surfaces from the value and normal gradient of vorticity on the surface (Ref. 8) and then the integrated forces and moments. In addition, the general formulas suggested several promising techniques for minimizing the required computation.

Several important theorems of fluid dynamics are utilized in the derivation of general formulas for aerodynamic forces and moments. Similar theorems and formulas are given in many well-known textbooks on aerodynamics, e.g. Refs. 9 and 10. These earlier theorems and formulas, however, are traditionally considered in the context of an infinite limitless fluid, i.e., an infinite fluid with no internal boundaries. The present theorems and formulas are valid in the presence of internal boundaries that represent solid surfaces. The practical importance of this more general validity needs no emphasis since the interaction between the solids and the fluid is indeed what the subject of aerodynamics is about.

The subject of this report received some attention in the 1950's, nearly a quarter of a century ago. Phillips (Ref. 11) presented a formula relating the fluid momentum to an integral of vorticity moment for a two-dimensional flow associated with a cylinder in translation. The general formulas presented in this report are valid for two- and three-dimensional flows associated with one or more finite solid bodies of any arbitrary shape executing any prescribed steady or time-dependent translation and/or rotation. Moreau (Ref. 12) presented formulas for a limitless fluid and for a portion of fluid subject to certain "order" conditions at infinity.

Truesdell (Ref. 13) commented that "Moreau emphasized his application to a limitless fluid, all but a finite interior part of which is in irrotational or circulation preserving motion. In this connection we should beware of the extremely strong order conditions at infinity required in order to get simple results, order conditions, indeed, which possibly may never be satisfied". In this report, formulas are rigorously derived for the viscous flow of fluids past finite bodies, using order conditions at infinity that are shown to be satisfied under quite general circumstances. The applications of the theory formed from these formulas will be presented in future reports.

II. VORTICITY DYNAMICS

The time-dependent motion of an infinite incompressible fluid with uniform viscosity relative to one or more immersed solid bodies is considered in the present study. The solid bodies are initially at rest in the fluid also at rest and are located within finite distances from one another. Subsequent prescribed motions of the solid bodies induce a corresponding motion of the fluid. At large time levels after the motion has initiated, if the solid bodies move uniformly at a constant translational velocity relative to the freestream, then the possibility of an asymptotic steady flow exists. Alternatively, the possibility of a time-dependent flow involving periodic vortex shedding, as evidenced by the well-known Karman vortex street behind a circular cylinder, also exists. In the present work, a steady flow, when it exists, is considered to be approached asymptotically at large time levels after the initiation of the solid motion. If the solid bodies do not move uniformly, or if the solid motion is time-dependent, then the motion of the fluid is necessarily time-dependent.

The familiar differential equations describing the time-dependent fluid motion are the continuity and Navier-Stokes equations:

$$\vec{\nabla}\cdot\vec{v} = 0 \tag{II-1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\vec{\nabla}^2\vec{v} \tag{II-2}$$

where $\vec{v}$, p, ρ, and ν are respectively the velocity vector, the pressure, the density, and the kinematic viscosity of the fluid, ρ being a constant in the present study. For simplicity, the kinematic viscosity of the fluid is considered to be uniform in this report. It is not difficult

to generalize the analyses given here to flows where the viscosity is not uniform. Such a generalization, however, is not essential to the purpose of the present work.

In this report, the region occupied by the fluid is designated R_f. A coordinate system with its origin located within finite distances from all solid surfaces, collectively designated by B_S, is used. Unless otherwise specified, this coordinate system is considered to be at rest relative to the freestream. The fluid region R_f is bounded internally by B_S and externally by a close boundary B_∞ at infinity. The region occupied by the j^{th} solid body is designated R_j, which is bounded externally by B_j. The limitless region jointly occupied by all the solid bodies and the fluid is designated R_∞.

It is convenient to introduce the vorticity vector $\vec{\omega}$ defined by

$$\vec{\nabla} \times \vec{v} = \vec{\omega} \tag{II-3}$$

and to consider the vorticity transport equation

$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega}) + \nu \nabla^2 \vec{\omega} \tag{II-4}$$

obtained by taking the curl of both sides of Eq. (II-2) and using Eqs. (II-1) and (II-3).

The set of equations (II-1), (II-3), and (II-4) replaces the set of equations (II-1) and (II-2). There are several reasons for using $\vec{\omega}$ in the formulation of the problem. In the first place, the circulation theory for the lift force suggests that the vorticity of the flow, which ultimately should be traced to the circulation, is responsible for the forces and moments exerted by the fluid on the solid. Secondly, as shown in Reference 6, the use of the vorticity vector, which is intimately connected

with viscosity effects, permits the solution field for the incompressible flow problem to be confined to the viscous region only. Thirdly, the set of equations in terms of $\vec{\omega}$ decomposes conveniently into a kinematic aspect and a kinetic aspect, each aspect constituting an entity by itself.

The first feature stated above provided the motivation for the present effort in developing general formulas relating aerodynamic forces and moments to the time-variation of vorticity-moment integrals. The advantages offered by the second feature have been extensively studied by this author and his coworkers in a series of previous articles in the context of computation methods. Most of the numerical results obtained thus far have been for two-dimensional incompressible laminar flows, both time-dependent (Refs. 6, 7, 8) and steady-state (Ref. 14). Some results have been obtained recently, however, for relatively simple turbulent flows (Ref. 15). In addition, extensions to compressible flow problems have been suggested (Refs. 16, 17).

The importance of the third feature in the present work is due to the fact that the physical processes of flow development are clearly delineated once the overall problem is decomposed into its kinematic and kinetic aspects. In particular, with vorticity as a field variable, considerable insight is gained by examining the differential equations describing these two aspects, without employing detailed mathematical or numerical analyses.

The kinematic aspect of the problem concerns the relationship between the vorticity distribution at any given instant of time and the velocity distribution at the same instant. The differential equations describing this aspect are the continuity and vorticity definition equations (II-1) and (II-4). Since the density of the solid bodies does not undergo appreciable change, the continuity equation (II-1) is obviously valid within the solid regions R_j as well as in the fluid region R_f. If, within a solid region R_j, one defines a vorticity field according to Eq. (II-3), then the kinematics

of the solid bodies and the fluid are described by the same differential equations. The stress-strain relations, which differentiate the fluid from the solid bodies kinetically, do not enter the kinematic relation betweeen the velocity field and the vorticity field. As a consequence, the solid bodies and the fluid can be treated together as one kinematical system.

The recognization of the fact just mentioned makes it relatively simple to derive the kinematic theorems and formulas presented in the report. For these theorems and formulas, the region of interest is limitless and the differential equations leading to these theorems and formulas are linear. In fact, all previously available kinematic theorems and formulas derived for an infinite unlimited fluid are immediately applicable to the present situation of an infinite fluid with one or more immersed solid bodies. This fact is not well recognized. The treatment of the solid bodies and the fluid together as one kinematical system has not received emphasis in the literature. In this report, kinematic theorems and formulas are derived by treating the solid bodies and the fluid together. The author has, in addition, re-derived each of these formulas and theorems by considering only the limited fluid region which is bounded internally by solid surfaces. The presence of boundaries makes the derivations lengthy and algebraically tedious. In this report, only a few of these re-derivations are presented to demonstrate the validity of treating the solid bodies and the fluid together as one kinematical system.

The kinetic aspect of the problem is concerned with the development of the vorticity field with time. This aspect is described by the vorticity transport equation (II-4). This equation is non-linear in the sense that the first term on its right-hand side involves the product of $\vec{v}$ and $\vec{\omega}$, and $\vec{v}$ is kinematically a function of $\vec{\omega}$. This equation is valid only in the fluid domain, which is limited.

Because of the non-linearity of the differential equation and the necessity of treating a limited region, the analysis of the kinetic aspect of the problem presents greater mathematical difficulties than the analysis of the kinematic aspect. Under certain conditions, it is possible to specify the vorticity field approximately without actually solving the vorticity transport equation. It is then only necessary to deal with the kinematic aspect of the problem which, as stated earlier, is described by linear differential equations applicable to the entire limitless region R_∞. Under these circumstances, the aerodynamic forces and moments are sometimes obtainable in a relatively simple manner.

The available literature on classical theories for aerodynamic forces and moments shows that the avoidance of the kinetic part of the problem has been an essential ingredient of these theories. There have been many recent efforts, via numerical methods, to treat the kinetic aspect either partially, e.g., on the basis of the inviscid flow equations, or fully. These efforts can benefit from a clear understanding of the physical processes involved in the development of the vorticity field in the fluid. A significant amount of information already exists in the literature on this topic (e.g. See Ref. 10). Those features of vorticity-field development that are pertinent to the present work are described briefly below.

For an inviscid fluid, the last term in Eq. (II-4) vanishes and the vorticity is convected with the fluid in the sense that the vorticity flux $\vec{\omega} \cdot d\vec{s}$ associated with each material element ds moving with the fluid remains a constant for all times. This well-known theorem of Helmholtz, a proof of which is available in many textbooks, e.g. Ref. 18, is modified in the case of a real fluid by the process of vorticity diffusion. According to Eq. (II-4), changes in the vorticity flux take place only by diffusion.

Vorticity flux cannot be created or destroyed in the interior of a fluid.

For the problem under consideration, the vorticity is obviously everywhere zero prior to the impulsive start of the motion of the solid bodies. The interior of the fluid domain therefore can become vortical only if vorticity diffuses across the boundaries of the fluid region. Consequently, immediately after the onset of the motion, the vorticity is everywhere zero in the fluid except at the boundaries in contact with the solid bodies. That is, the fluid motion immediately after the onset of the motion has a non-zero tangential velocity relative to the solid bodies at the solid boundaries. The discontinuity in tangential velocity constitutes a sheet of concentrated vorticity (vortex sheet) at the boundaries. At subsequent time levels, this concentrated vorticity spreads into the interior of the fluid domain by diffusion and, once there, is transported away from the boundaries by both convection and diffusion. At the same time, the no-slip condition provides a mechanism for the continual generation of vorticity at the boundaries. The general flow pattern therefore contains vortical regions surrounding the solid bodies and vortical wakes trailing the solid bodies. Outside of these vortical regions and wakes, the flow is essentially free of vorticity and therefore irrotational. In particular, if the flow Reynolds number is not small, then the vorticity spreads by diffusion only a short distance from the boundaries before it is carried away with the fluid by convection. Therefore a large region of the fluid, ahead and to the side of the solid bodies, is essentially free of vorticity and irrotational.

In the next two Chapters, a number of theorems and formulas are derived using the above described kinematic and kinetic characteristics of the flow.

III. SELECTED THEOREMS AND FORMULAS FOR THE KINETIC ASPECT

1. Asymptotic Behavior and Effective Extent of Vorticity Field.

For the present problem, the vorticity decays exponentially with increasing distance from the origin at large distances for all finite time levels after the onset of the motion and the effective extend of the vortical region is finite. The finite extent of the vortical region is a consequence of the fact that the vorticity is transported in the fluid by finite-rate processes and cannot be created in the interior of a fluid domain. This fact as well as the asymptotic behavior of the vorticity distribution at large distances from solid bodies are established below by considering the fundamental solution of the diffusion equation.

The fundamental solution F of the diffusion equation, i.e. the Green's function for an infinite unlimited region, can be expressed as (Ref. 18 and 19)

$$F(\vec{r}, t; \vec{r}_o, t_o) = \frac{1}{(4\pi\nu(t-t_o))^{\frac{d}{2}}} \exp\left\{ - \frac{|\vec{r}-\vec{r}_o|^2}{4\,\nu(t-t_o)} \right\} \qquad \text{(III-1)}$$

where $\vec{r}$ and $\vec{r}_o$ are position vectors and d is the dimensionality of the problem, i.e., d = 1, 2, and 3 respectively for one-, two-, and three-dimensional problems. For the present problem, the fundamental solution represents the vorticity distribution at the time level t resulting from the diffusion of a concentrated vorticity which is of "unit" strength and is located at the point $\vec{r}_o$ at the time level t_o, with $t_o < t$.

If, at the time level t_o, the vorticity is non-zero only in an elemental region dR_o located at $\vec{r}_o$ and the value of vorticity in dR_o is $\vec{\omega}_o$, then the vorticity distribution at the subsequent time level t is $F\vec{\omega}_o dR_o$. If, at the time level t_o, the vorticity distribution is known in the unlimited region R_∞, then the vorticity distribution at a subsequent time level t is expressible as an integral:

$$\vec{\omega}(\vec{r}, t) = \int_{R_\infty} F\vec{\omega}(\vec{r}_o, t_o)dR_o \qquad \text{(III-2)}$$

where the subscript for dR_o indicates that the integration is performed in the $\vec{r}_o$ space.

Equation (III-2) is valid in an infinite unlimited region in the absence of convection. The form of the fundamental solution F clearly shows that, if the vorticity is non-zero at the time level t_o only within finite distances from the origin, then the vorticity at any subsequent finite time level t approaches zero exponentially with increasing distance r from the origin, at large distances. Therefore, the vortical region is effectively confined to a finite region at any finite time level t.

For a fluid region bounded internally by solid surfaces and in which convective process is present, Eq.(III-2) needs to be generalized. It is obvious that the convective process, being one of finite rate, does not alter the above conclusions regarding the effective extent of vortical regions and the asymptotic behavior of vorticity at any finite time level. Similarly, the presence of solid surfaces in the fluid provides a mechanism for introducing vorticity at the boundaries of the fluid region and, as long as these boundaries are within finite distances from the origin, the introduction of vorticity does not alter the above conclusions. For

two-dimensional flows, the generalized version of Eq. (III-2) is expressible as (Refs. 17 and 20):

$$\omega(\vec{r}, t) = \int_{R_f} (F\,\omega_o)_{t_o=0}\, dR_o$$

$$+ \int_0^t dt_o \int_{R_f} \omega_o \vec{v}_o \cdot \vec{\nabla}_o F dR_o + \nu \int_0^t dt_o \oint_{B_S} (F\, \vec{\nabla}_o \omega_o - \omega_o \vec{\nabla}_o F) \cdot \vec{n}_o dB_o \qquad \text{(III-3)}$$

where the subscript "o" indicates that the variables, differentiations, and integrations are in the $\vec{r}_o$, t_o space, e.g. $\omega_o = \omega_o(\vec{r}_o, t_o)$. The second and third integrals of Eq. (III-3) represent respectively the convective process and the effect of solid boundaries. Each of the integrands in the integrals of Eq. (III-3) is directly proportional to F and/or $\vec{\nabla} F$. At any given time level t and at large distances r from the origin, F and $\vec{\nabla} F$ decays exponentially with increasing r. It follows that, for any problem in which B_S consists of finite surfaces located within finite distances from the origin, if the vorticity decays exponential with increasing r at large r for all time levels previous to t, then the vorticity decays exponentially with increasing r also at the time level t. In fact, for such a problem, if the vorticity is confined within finite distances from the origin at any given instant of time, then the vorticity decays exponentially with increasing r at large r at all subsequent finite time levels. For the present problem involving an impulsively started motion, the vorticity is non-zero immediately after the onset of the motion only on the solid surfaces. Consequently, the statements made in the first paragraph of this Section are true for two-dimensional flows. It can be similarly shown that these statements are also true for three-dimensional flows.

2. Principle of Total Vorticity Conservation -- Two-Dimensional Flows.

It shall be shown that, for the flow of an incompressible fluid past solid bodies, the total vorticity in the infinite unlimited space occupied jointly by the fluid and the solid bodies is invariant with respect to time, provided that an order condition for the vorticity at infinity is satisfied. This order condition is that the vorticity approaches zero as r^{-n}, where $n > d$, d being the dimensionality of the problem.

The above statement will be referred to as the principle of total vorticity conservation and is expressible mathematically as

$$\frac{d}{dt} \int_{R_\infty} \vec{\omega} \, dR = 0 \qquad \text{(III-4)}$$

provided that ω approaches r^{-n} as $r \to \infty$, with $n > d$.

As noted earlier, the solid bodies and the fluid can be treated together as one kinematic system in the present problem. For three-dimensional flows, since the vorticity field is solenoidal and is effectively confined to a finite region, all vorticity-lines in the combined system form closed curves. Consequently, one has

$$\int_{R_\infty} \vec{\omega} \, dR = 0 \qquad \text{(III-5)}$$

In the next Chapter, a proof of Eq. (III-5) is presented treating the solid bodies and the fluid as separate kinematical systems. Clearly, Eq. (III-4) follows directly from Eq. (III-5). Thus the total vorticity is not only conserved, it must always be zero in three-dimensional flows.

For two-dimensional flows, the vorticity-lines are directed perpendicular to the plane of the flow. The vorticity-lines extend to infinity in the

direction perpendicular to the flow, and they do not form closed curves in the plane of the flow. The total vorticity of the fluid and the solid bodies, that is, the integral of vorticity in the infinite unlimited plane of the flow, is still conserved. This principle of total vorticity conservation for two-dimensional flows is not usually discussed in standard treatises on fluid dynamics. A proof of this principle for two-dimensional flows is given below on the basis of the kinetics of the problem.

For two-dimensional flows, the vorticity transport equation (II-4) can be rewritten as

$$\frac{D\vec{\omega}}{Dt} = -\nu\vec{\nabla} \times \vec{\nabla} \times \vec{\omega} \qquad \text{(III-6)}$$

where $\frac{D}{Dt}$ denotes a substantial derivative.

The time rate of change of the total vorticity of the fluid is

$$\frac{d}{dt}\int_{R_f(t)} \vec{\omega}\, dR = \int_{R_f(t)} \frac{D\vec{\omega}}{Dt}\, dR \qquad \text{(III-7)}$$

where $R_f(t)$ is the entire region occupied by the fluid and is a function of the time.

Placing Eq. (III-6) into Eq. (III-7) and using Stoke's theorem, one obtains

$$\frac{d}{dt}\int_{R_f(t)} \vec{\omega}\, dR = \nu \int_{B(t)} (\vec{\nabla} \times \vec{\omega}) \times \vec{n}\, dB \qquad \text{(III-8)}$$

where $\vec{n}$ is a unit outward directed normal vector.

In Eq. (III-8), the boundary B consists of the solid boundary B_s and the boundary at infinity B_∞. The contribution of B_∞ to Eq.(III 5)

is zero provided that ω approach zero as r^{-n} for large r, with $n > 2$. Since ω approaches zero exponentially with increasing r for large r, the above order condition at large r is satisfied for the present problem. The boundary B in Eq. (III-8) can therefore be replaced by the solid boundary B_S.

Equation (II-2) can be rewritten as

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\vec{\nabla}p - \nu\vec{\nabla}\times\vec{\omega} \tag{III-9}$$

Taking the vector product of each term in Eq. (III-9) with $\vec{n}$ and integrate the resulting equation around B_S, one has

$$\oint_{B_S}\frac{D\vec{v}}{Dt}\times\vec{n}dB = -\frac{1}{\rho}\oint_{B_S}\vec{\nabla}p\times\vec{n}dB - \nu\oint_{B_S}(\vec{\nabla}\times\vec{\omega})\times\vec{n}dB \tag{III-10}$$

The first integral on the right-hand side of Eq. (III-10) is zero by virtue of the single-valuedness of pressure on B_S. This fact can also be shown by using the Stoke's theorem and the fact that the curl of the gradient of any scalar function is zero. Combining Eqs. (III-8) and (III-10) therefore gives

$$\frac{d}{dt}\int_{R_f(t)}\vec{\omega}\,dR = -\oint_{B_S}\frac{D\vec{v}}{Dt}\times\vec{n}dB \tag{III-11}$$

Consider now the region R_S bound externally by B_S. With the no-slip condition, the substantial acceleration $\frac{D\vec{v}}{Dt}$ on B_S is identical for the solid bodies and for the fluid. Using Stokes's theorem and the fact that the outward normal vector for R_S is directed opposite to that for R_f, one obtains:

$$\frac{d}{dt}\int_{R_f(t)}\vec{\omega}\,dR = -\int_{R_S(t)}\vec{\nabla}\times\left(\frac{D\vec{v}}{Dt}\right)dR \tag{III-12}$$

In the solid region R_j, the velocity vector is given by

$$\vec{v} = \vec{v}_j + \vec{\Omega}_j \times \vec{r} \qquad \text{(III-13)}$$

where $\vec{v}_j$ is the rectilinear velocity of the solid body j and $\vec{\Omega}_j$ is its rotational velocity. For two-dimensional problems, $\vec{\Omega}_j$ is directed perpendicular to the plane of the flow. The vorticity in the solid body j is readily obtainable by taking the curl of the velocity vector as given by Eq. (III-13) and is

$$\vec{\omega} = 2\vec{\Omega}_j \qquad \text{(III-14)}$$

Using vector differential identities, it is simple to show that $\vec{\nabla}\times(\frac{D\vec{v}}{Dt}) = \frac{D\vec{\omega}}{Dt}$ in each of the solid regions R_j. One thus obtains from Eq. (III-12)

$$\frac{d}{dt}\int_{R_f(t)} \vec{\omega}\, dR = -\frac{d}{dt}\int_{R_S(t)} \vec{\omega}\, dR \qquad \text{(III-15)}$$

Equation (III-15) is equivalent to Eq. (III-4). It states that the rate of change of the total vorticity is equal in magnitude and opposite in sign to the rate of change of the total vorticity in the solid bodies, or, equivalently, the total vorticity in the infinite unlimited region R_∞ is zero. If the rotational velocities $\vec{\Omega}_j$ of the solid bodies are prescribed functions of time, then the rate of change of the total vorticity in the fluid can be calculated using the following simple formula:

$$\frac{d}{dt}\int_{R_f} \vec{\omega}\, dR = -2\sum_{j=1}^{N} \frac{d\,\Omega_j}{dt} R_j \qquad \text{(III-16)}$$

where N is the total number of solid bodies present, and R_j is the size of the solid body j.

Equation (III-15) can be integrated with respect to time, yielding

$$\int_{R_f(t)} \vec{\omega}\, dR + \int_{R_S(t)} \vec{\omega}\, dR = \vec{A} \qquad \text{(III-17)}$$

where $\vec{A}$ is a constant vector.

For a motion starting from rest, the total vorticity in the combined fluid and solid regions is zero before the onset of the motion. Consequently, $\vec{A} = 0$ and one obtains (III-5), which states that the vorticity in the combined solid and fluid regions, i.e., the infinite unlimited region, is always zero. Thus, if the solid motion is prescribed at any time level, the total vorticity in the fluid is easily calculated from

$$\int_{R_f} \vec{\omega}\, dR = -2 \sum_{j=1}^{N} \vec{\Omega}_j R_j \qquad \text{(III-18)}$$

There are several conceptual differences between the principle of total vorticity conservation discussed here and the usual understanding of invariance of vorticity integral with respect to time.

In the present work, the solid regions are included in the evaluation of the total vorticity. The integrands of Eqs. (III-4) and (III-5) are piecewise continuous and the integrals converge. The meaning of the total vorticity in the unlimited infinite space jointly occupied by the fluid and the solid bodies is unambiguous. While previous discussions of the invariance of the total vorticity are usually made in the context of an inviscid fluid or of an unlimited infinite fluid region (in the absence of internal boundaries), the present study utilizes the no-slip condition at the solid boundaries and permits the presence of such boundaries in

the fluid. In Ref. 21, Section B.2, a formula similar to Eq. (III-18), but specialized to a single solid and including an additional term representing the contribution of the velocity at a surface enclosing the solid, is presented. In the present work, it is shown that this contribution is absent if the surface is sufficiently distant from the solid bodies and the order condition for the vorticity vector is satisfied. The formula (III-18) is derived for one or more solid bodies in the present Chapter.

Ref. 21 also gives a formula for the rate of change of the total vorticity in the fluid. That formula contains a term representing "conduction of vorticity through the solid boundary." It is pointed out that further dynamic (kinetic) equations are needed to evaluate this term. The present result, Eq. (III-16), shows that this term is given simply by the rate of change of the total vorticity of the solid bodies.

In Ref. 10, it is shown that in three-dimensions the total vorticity is zero in a region which contains the fluid region and "a region extending beyond the actual boundaries". The present results show that the proper extension of the fluid region is simply the solid regions in which the correct vorticity values to assign are the actual vorticity of the solid bodies. For two-dimensional flows, the literature emphasizes the possibility of the existence of a non-zero circulation around closed paths at large distances from the solids. According to the present results, this possibility does not exist for a real fluid at any finite time level after a motion has started from rest. An evaluation of the conduction of vorticity through solid boundaries is not necessary since this "conduction process" conserves the total vorticity in the infinite unlimited region occupied jointly by the fluid and the solid bodies.

3. Stress Outside the Vortical Regions.

Outside the vortical regions, the vorticity is zero and the viscous stress is absent. The momentum equation (II-2) simplifies to

$$\rho \frac{\partial \vec{v}}{\partial t} = - \vec{\nabla}(p + \rho v^2/2) \qquad \text{(III-19)}$$

The absence of vorticity implies the existence of a scalar potential Φ such that

$$\vec{v} = -\vec{\nabla}\Phi \qquad \text{(III-20)}$$

Placing Eq. (III-20) into Eq. (III-19) and integrating the resulting equation in space gives

$$p = \rho \frac{\partial \Phi}{\partial t} - \frac{\rho v^2}{2} + f(t) \qquad \text{(III-21)}$$

This well-known equation for unsteady inviscid pressure will be utilized to derive general formulas for aerodynamic forces and moments in Chapter V. It should be noted that the scalar potential is single-valued. For three-dimensional flows past finite solid bodies, the region in which Φ exists is singly-connected and therefore Φ is independent of path. For two-dimensional flows, the region in which Φ exists is multiply-connected. However, the cyclic constant for Φ is zero since the vortical regions are of finite extent and the total vorticity is zero.

IV. SELECTED THEOREMS AND FORMULAS FOR THE KINEMATIC ASPECT

The proofs of the theorems and formulas for the kinematic aspect of the external flow problem utilize only the kinematic relationship between the velocity and the vorticity fields and the order conditions for the vorticity field.

1. <u>Principle of Total Vorticity Conservation--Three-Dimensional Flows.</u>

Let $\vec{n}$, $\vec{b}_1$, and $\vec{b}_2$ be a right-handed set of orthogonal vectors on the boundary B_S. $\vec{b}_1$, $\vec{b}_2$ are tangential unit vectors. Let the velocity components in the $\vec{n}$, $\vec{b}_1$, and $\vec{b}_2$ directions be v_n, v_{b1}, and v_{b2} respectively. With the no-slip condition, these velocity components are identical on B_S for the fluid and for the solid bodies. The normal component of the vorticity vector on the boundary B_S is given by $\frac{\partial v_{b_2}}{\partial b_1} - \frac{\partial v_{b_1}}{\partial b_2}$ and is continuous across B_S. That is $\vec{\omega} \,.\, \vec{n}$ on B_S for the fluid is identical to that for the solid body.

Using the fact that $\vec{\omega}$ is solenoidal, one obtains through vector differential identities:

$$\vec{\nabla} \cdot \left((\vec{a} . \vec{r}) \, \vec{\omega} \right) = \vec{a} . \vec{\omega} \tag{IV-1}$$

where $\vec{a}$ is an arbitrary constant vector.

The divergence theorem then gives

$$\int_R \vec{\omega} \, dR = \oint_B \vec{r} (\vec{\omega} \cdot \vec{n}) \, dB \tag{IV-2}$$

Let R be the fluid region R_f. The boundary B then consists of the solid boundary B_S and a boundary at infinity. The contribution of the boundary at infinity is zero provided that ω approaches zero as r^{-n} for large r, with $n > 3$. This order condition is certainly met in the present problem for which ω decays exponentially with r for large r. Therefore, the boundary B in Eq. (IV-2) can be replaced by B_S.

Since the normal component of $\vec{\omega}$ is continuous across B_S, one may replace $\vec{\omega}\cdot\vec{n}$ in the right-hand side of Eq. (IV-2) by $2\,\vec{\Omega}_j\cdot\vec{n}$, where $\vec{\Omega}_j$ is the angular velocity of the j^{th} solid body. One then has

$$\int_{R_f} \vec{\omega}\, dR = 2 \sum_{j=1}^{N} \oint_{B_j} \vec{r}(\vec{\Omega}_j \cdot \vec{n}) dB \tag{IV-3}$$

Consider now the region R in Eq. (III-2) to be the region R_j bounded externally by B_j. One has

$$2 \int_{B_j} \vec{r}(\vec{\Omega}_j \cdot \vec{n}) dB = 2 \int_{R_j} \vec{\Omega}_j dR \tag{IV-4}$$

Placing Eq. (IV-4) into Eq. (IV-3) and noting that the normal vector $\vec{n}$ in Eq. (IV-3) is directed outward from R_f while it is directed inward from R_f in Eq. (IV-4), one obtains

$$\int_{R_f} \vec{\omega}\, dR = -2 \sum_{j} \vec{\Omega}_j R_j \tag{IV-5}$$

which is equivalent to Eq. (III-5).

2. Biot-Savart's Law.

It is shown in Ref. 6 that, by using the fundamental solution of the Poisson's equation, the kinematics of the problem, i.e. Eqs. (II-1) and (II-3), is expressible in the form of an integral representation:

$$\vec{v}(\vec{r}, t) = -\frac{1}{A} \int_R \frac{\vec{\omega}_o \times (\vec{r}_o - \vec{r})}{|\vec{r}_o - \vec{r}|^d} \, dR_o$$

$$+ \frac{1}{A} \int_B \frac{(\vec{v}_o \cdot \vec{n}_o)(\vec{r}_o - \vec{r}) - (\vec{v}_o \times \vec{n}_o) \times (\vec{r}_o - \vec{r})}{|r_o - r|^d} \, dB_o \qquad \text{(IV-6)}$$

where A and d are constants depending on the dimensionality of the problem --- $A = 4\pi$ and $d = 3$ for three-dimensional problems, $A = 2\pi$ and $d = 2$ for two-dimensional problems---and the subscript "o" indicates that the variables and the integrations are in the $\vec{r}_o$ space, i.e. $\vec{\omega}_o = \vec{\omega}(\vec{r}_o, t)$ etc.

Let the region R be the fluid region R_f bounded internally by the solid boundary B_s and externally by a boundary at infinity. With a coordinate system attached to the freestream, the velocity at infinity is zero and the contribution of the boundary integral to the velocity field is therefore limited to the solid boundary B_s.

The contribution of the solid boundary B_j to the velocity field is expressible as

$$\vec{I}_j = \oint_{B_j} \left[(\vec{v}_o \cdot \vec{n}_o) \vec{\nabla} P - (\vec{v}_o \times \vec{n}_o) \times \vec{\nabla} P \right] dB_o \qquad \text{(IV-7)}$$

where P is the fundamental solution of the Poisson's equation and is defined by

$$P = \begin{cases} \dfrac{1}{4\pi |r - r_o|} & \text{for three-dimensional problems} \\ \dfrac{1}{2\pi} \ln \dfrac{1}{|r - r_o|} & \text{for two-dimensional problems} \end{cases}$$

Using vector identities, Eq. (IV-7) can be rewritten as

$$\vec{I}_j = \vec{\nabla} \oint_{B_j} P\vec{v}_o \cdot \vec{n}_o dB_o + \vec{\nabla} \times \oint_{B_j} P\vec{v}_o \times \vec{n}_o dB_o \qquad \text{(IV-8)}$$

Consider now the solid region R_j bounded externally by B_j. Using the divergence theorem and Stoke's theorem, and noting that the outward normal for the region R_j is directed opposite to $\vec{n}$, one obtains from Eq. (IV-8)

$$\vec{I}_j = \int_{R_j} \left[-\vec{\nabla}\left(\vec{\nabla}_o \cdot (P\,\vec{v}_o)\right) + \vec{\nabla} \times \vec{\nabla}_o \times (P\,\vec{v}_o) \right] dR_o \qquad \text{(IV-9)}$$

Using vector differential identities, the integrand in Eq. (IV-9) can be re-expressed as

$$-(\vec{\nabla}_o \times \vec{v}_o) \times \vec{\nabla} P - \vec{v}_o(\vec{\nabla} \cdot \vec{\nabla}_o P) \qquad \text{(IV-10)}$$

It is easy to show that

$$\vec{\nabla} \cdot \vec{\nabla}_o P = -\nabla^2 P = 0 \text{ for } \vec{r} \neq \vec{r}_o$$

Thus, in the fluid region R_f, one has,

$$I_j = -\frac{2}{A} \int_{R_j} \frac{\vec{\Omega}_j \times (\vec{r}_o - \vec{r})}{|\vec{r}_o - \vec{r}|^d} dR_o$$

One therefore obtains from Eq. (IV-6),

$$\vec{v}(\vec{r}, t) = -\frac{1}{A} \int_{R_\infty} \frac{\vec{\omega}_o \times (\vec{r}_o - \vec{r})}{|r_o - r|^d} dR_o \qquad \text{(IV-11)}$$

where R_∞ is the unlimited infinite region jointly occupied by the fluid and the solid bodies. Equation (IV-11) is an expression of the Biot-Savart's law for a distribution of vorticity in an unlimited infinite region. In fact, by considering the solid bodies and the fluid together as one kinematical system, Eq. (IV-11) can be immediately written down without going through the intermediate steps dealing with the boundary conditions on B_S.

3. Asymptotic Behavior of Velocity Field.

To examine the asymptotic behavior of the velocity field at large distances from the solid bodies, one re-expresses Eq. (IV-11) as

$$\vec{v}(\vec{r}, t) = \vec{\nabla} \times \int_{R_\infty} \vec{\omega}_o P dR_o \qquad \text{(IV-12)}$$

For three-dimensional problems, expressing the function $\frac{1}{|r_o - r|}$ as a Taylor series about the point $\vec{r}_o = 0$, one obtains

$$\frac{1}{|\vec{r}_o - \vec{r}|} = \frac{1}{|\vec{r}_o - \vec{r}|}\Big|_{r_o=0} + \vec{r}_o \cdot \left[\left(\vec{\nabla}_o \frac{1}{|\vec{r}_o - \vec{r}|}\right)\Big|_{r_o=0}\right]$$

$$+ \frac{1}{2}\vec{r}_o\vec{r}_o : \left[\left(\vec{\nabla}_o \vec{\nabla}_o \frac{1}{|\vec{r}_o - \vec{r}|}\right)\Big|_{r_o=0}\right] + \ldots$$

Or

$$\frac{1}{|r_o - r|} = \frac{1}{r} - \vec{r}_o \cdot \vec{\nabla}\frac{1}{r} + \text{terms of order } r^{-n}, \; n \geq 3 \qquad \text{(IV-13)}$$

One has from Eqs. (IV-8), (IV-12), and (IV-13), the following expression for the velocity vector

$$\vec{v}(\vec{r}, t) = \frac{1}{2\pi} (\vec{\nabla} \frac{1}{r}) \times \int_{R_\infty} \vec{\omega}_o dR_o - \frac{1}{4\pi} \vec{\nabla} \times \int_{R_\infty} (\vec{r}_o . \vec{\nabla} \frac{1}{r}) \vec{\omega}_o dR_o \qquad \text{(IV-14)}$$

$$+ \text{ terms of order } r^{-n}, \quad n \geq 4$$

The first term on the right-hand side of Eq. (IV-4) is zero because the total vorticity in the infinite unlimited region is zero, i.e. Eq.(IV-5) or Eq. (III-5).

Using the identity

$$\nabla_o \left[(r_o \vec{\nabla} \frac{1}{r}) \vec{\omega}_o \vec{r}_o \right] = (\vec{\nabla} \frac{1}{r} \vec{\omega}_o) \vec{r}_o + (\vec{\nabla} \frac{1}{r} . \vec{r}_o) \vec{\omega}_o \qquad \text{(IV-15)}$$

the second term on the right-hand side of Eq. (IV-14) is rewritten, with the help of the divergence theorem, as

$$-\frac{1}{4\pi} \vec{\nabla} \times \left\{ \frac{1}{2} \int_R \left[(\vec{\nabla} \frac{1}{r} . r_o) \vec{\omega}_o - (\vec{\nabla} \frac{1}{r} . \vec{\omega}_o) \vec{r}_o \right] dR_o \right. \qquad \text{(IV-16)}$$

$$\left. - \frac{1}{2} \int_{B_\infty} (\vec{r}_o \vec{\nabla} \frac{1}{r}) \vec{r}_o (\vec{\omega}_o . \vec{n}_o) dB_o \right\}$$

The last integral in the above expression vanishes if ω approaches zero as r^{-n} for large r, with $n > 4$. This condition is certainly met, as discussed in Chapter III. The remainder of the above expression can be rewritten as

$$\frac{1}{4\pi} \vec{\nabla} \times (\vec{\nabla} \frac{1}{r} \times \vec{\alpha}_o) \qquad \text{(IV-17)}$$

or $\quad \frac{1}{4\pi} \vec{\nabla} (\vec{\nabla} \frac{1}{r} . \vec{\alpha}_o)$

where $\vec{\alpha}_o$ is the total first moment of the vorticity $\vec{\omega}_o$ defined by

$$\vec{\alpha}_o = \int_{R_\infty} \vec{r}_o \times \vec{\omega}_o dR_o \tag{IV-18}$$

Equation (IV-14) is thus expressible as

$$\vec{v}(\vec{r}, t) = \frac{1}{8\pi}\vec{\nabla}(\vec{\nabla}\frac{1}{r} \cdot \vec{\alpha}_o) + \text{terms of order } r^{-n},\ n \geq 4 \tag{IV-19}$$

The first term on the righthand side of Eq. (IV-18) is of order r^{-3}. Provided that $\vec{\alpha}_o$ is not zero, $\vec{v}$ is of order r^{-3} for large r.

For two-dimensional problems, the function In $\frac{1}{|r_o - r|}$ can be expressed as a Taylor series about the point $r_o = 0$:

$$\ln \frac{1}{|r_o - r|} = \ln \frac{1}{r} - \vec{r}_o . \vec{\nabla} \ln \frac{1}{r} + \text{terms of order } r^{-n},\ n \geq 3$$

Following similar procedures as that given above for three-dimensional problems, one obtains for two-dimensional problems:

$$\vec{v}(\vec{r}, t) = \frac{1}{2\pi}(\vec{\nabla} \ln\frac{1}{r}) \times \int_{R_\infty} \vec{\omega}_o dR_o - \frac{1}{2}\vec{\nabla} \times \int_{R_\infty} (\vec{r}_o . \vec{\nabla} \ln \frac{1}{r})\, \vec{\omega}_o dR_o$$

$$+ \text{terms of order } r^{-n},\ n > 3 \tag{IV-20}$$

The first term on the right-hand side of Eq. (IV-20) vanishes because of Eq. (III-18). The second term may be rewritten, by noting that

$$\vec{\omega}_o(\vec{r}_o . \vec{\nabla} \ln \frac{1}{r}) - \vec{r}_o(\vec{\omega}_o . \vec{\nabla} \ln \frac{1}{r}) = -(\vec{\nabla} \ln \frac{1}{r}) \times (\vec{r}_o \times \vec{\omega}_o)$$

and that in two-dimensional problems

$$\vec{\omega}_o \cdot \vec{\nabla} \ln \frac{1}{r} = 0$$

as

$$\frac{1}{2\pi}\vec{\nabla} \times (\vec{\nabla} \ln \frac{1}{r} \times \vec{\alpha}_o) = -\frac{1}{2\pi}(\nabla^2 \ln \frac{1}{r})\vec{\alpha}_o + \frac{1}{2\pi}(\vec{\alpha}_o \cdot \vec{\nabla})(\vec{\nabla} \ln \frac{1}{r})$$

The first term on the righthand side of the above expression vanishes for $r \neq 0$. One thus has, for two-dimensional problems

$$\vec{v}(\vec{r}, t) = \frac{1}{2\pi}\vec{\nabla}(\vec{\nabla} \ln \frac{1}{r} \cdot \vec{\alpha}_o) + \text{terms of order } r^{-n}, n \geq 3 \qquad \text{(IV-21)}$$

Therefore, provided that $\vec{\alpha}_o$ is not zero, $\vec{v}$ is of order r^{-2} for large r.

The above conclusions for two-dimensional and three-dimensional problems can be combined into the statement that, provided that $\vec{\alpha}_o$ is not zero, $\vec{v}$ *is of order* r^{-d} *for large* r, d *being the dimensionality of the* problem.

4. Velocity Integrals in Large Bounded Regions

Using Eqs. (II-1) and (II-3) and vector differential identities, one has, for three-dimensional problems, $\vec{\nabla}(\vec{r}\cdot\vec{v}) - \vec{\nabla}\cdot(\vec{r}\vec{v}) = \vec{r}\times\vec{\omega} - 2\vec{v}$.

The divergence theorem and its corollaries then give

$$\int_R (\vec{r} \times \vec{\omega} - 2\vec{v})dR = \oint_B \left[(\vec{r}\cdot\vec{v})\vec{n} - (\vec{n}\cdot\vec{r})\vec{v}\right] dB \qquad \text{(IV-22)}$$

The above equation can be re-expressed as

$$\int_R \vec{v}dR = \frac{1}{2}\int_R \vec{r} \times \vec{\omega}\, dR - \frac{1}{2}\oint_B \vec{r} \times (\vec{n} \times \vec{v})dB \qquad \text{(IV-23)}$$

Let R be the region R_j occupied by the solid body j. Equation (IV-23) gives

$$\frac{1}{2} \oint_{B_j} \vec{r} \times (\vec{n} \times \vec{v}) dB = \int_{R_j} \vec{r} \times \vec{\Omega}_j dR - \int_{R_j} \vec{v}_j dR \qquad \text{(IV-24)}$$

Consider now R in Eq. (IV-23) to be R'_f, the part of fluid region bounded internally by the solid surface B_S and externally by a large but finite spherical surface $r = L$. Let this spherical surface be designated B_L. Let L be sufficiently large so that B_L encloses all the solid bodies present. The boundary B in Eq. (IV-23) now consists of the solid boundary B_S and the spherical boundary B_L. With the no-slip condition, the contribution of B_S to Eq. (IV-23) is given by Eq. (IV-24). One then has, noting that for R'_f the normal vector $\vec{n}$ is directed into the solid where as in Eq. (IV-24) $\vec{n}$ is directed into the fluid,

$$\int_{R_f'} \vec{v} dR = -\sum_{j=1}^{N} \int_{R_j} \vec{v}_j dR + \frac{1}{2} \int_{R'_f} (\vec{r} \times \vec{\omega}) dR + \sum_{j=1}^{N} \int_{R_j} \vec{r} \times \vec{\Omega}_j dR$$

$$- \frac{1}{2} \oint_{B_L} \vec{r} \times (\vec{n} \times \vec{v}) dB \qquad \text{(IV-25)}$$

Equation (IV-25) can be re-expressed as

$$\int_{R_L} \vec{v} dR = \frac{1}{2} \int_{R_L} (\vec{r} \times \vec{\omega}) dR - \frac{1}{2} \oint_{B_L} \vec{r} \times (\vec{n} \times \vec{v}) dB \qquad \text{(IV-26)}$$

where R_L is the spherical region $r \leq L$ and includes the fluid region R_f' as well as all the solid regions R_j.

Let L be sufficiently large so that the total vorticity outside R_L is negligible and the velocity on B_L is accurately given by the first term on the right-hand side of Eq. (IV-19). The integral over the surface B_L in Eq. (IV-26) is finite and expressible in terms of the vorticity-moment integral $\vec{\alpha}$ as shown below.

Let $\vec{i}$, $\vec{j}$, and $\vec{k}$ be a set of right-handed normal unit vectors in a Cartesian coordinate system (x, y, z), in the directions x, y, and z respectively. The corresponding spherical coordinates (r, θ, ϕ) with unit vectors $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\phi$ are related to x, y, and z by

$$x = r \operatorname{Sin}\theta \operatorname{Cos}\phi$$

$$y = r \operatorname{Sin}\theta \operatorname{Sin}\phi$$

$$z = r \operatorname{Cos}\theta$$

On B_L, the first term of Eq. (IV-19) gives a velocity vector

$$\vec{v} = \frac{\alpha_o}{8\pi L^3} \left(2 \operatorname{Cos}\theta \vec{e}_r + \operatorname{Sin}\theta \vec{e}_\theta\right)$$

Also, on B_L. one has $r = L\vec{e}_r$ and $\vec{n} = \vec{e}_r$.

One thus has

$$\vec{r} \times (\vec{n} \times \vec{v}) = \frac{\alpha_o \operatorname{Sin}\theta}{8\pi L^2} \vec{e}_\theta$$

$$= \frac{\alpha_o}{8\pi L^2} \left(\operatorname{Sin}\theta \operatorname{Cos}\theta \operatorname{Cos}\phi \vec{i} + \operatorname{Sin}\theta \operatorname{Cos}\theta \operatorname{Sin}\phi \vec{j} - \operatorname{Sin}^2\theta \vec{k} \right)$$

Placing the above expression into the surface integral of Eq. (IV-26), one obtains $\frac{1}{3} \vec{\alpha}_o$ as the value of that integral. Noting that the first integral on the right-hand side of Eq. (IV-26) is $\vec{\alpha}$, one has

$$\int_{R_L} \vec{v}dR = \frac{1}{3}\vec{\alpha}_o \qquad \text{(IV-27)}$$

It should be pointed out that Eq. (IV-27) is valid for any sufficiently large but finite value of L. Although Eq. (IV-26) is independent of the size of R_L, one may not consider L to be infinitely large. The integral $\int_{R\infty} \vec{v}dR$ is in fact indeterminate if $\alpha \neq 0$. For example, the z-component of this integral over the region $r \geq L$ is, according to the first term of Eq. (IV-19),

$$\frac{\alpha}{4\pi} \int_{\Phi=0}^{2\pi} \int_{\Theta=0}^{\pi} \int_{r=L}^{\infty} \frac{1}{r} (2\text{Cos}^2\Theta - \text{Sin}^2\Theta)\text{Sin}\,\Theta\, dr d\Theta\, d\Phi$$

This integral gives

$$\frac{\alpha}{2} (\text{Cos}\Theta - \text{Cos}^3\Theta) \Big|_{\Theta=0}^{\pi} \ln(r) \Big|_{r=L}^{\infty}$$

which is indeterminate.

It is simple to show that the integrals of the x- and y-components of $\vec{v}$ over the infinite region $r \geq L$ are also indeterminate. Also, the integrals of the velocity moment $\vec{r} \times \vec{v}$ over the infinite region $r \geq L$ are indeterminate. To show that these latter integrals do not diverge, terms of order r^{-4} in Eq. (IV-19) must be considered.

Eq. (IV-22) is valid only for three-dimensional problems. For two-dimensional problems, one has, instead of Eq. (IV-22),

The two-dimensional version of Eq. (IV-26) is therefore

$$\int_{R_L} \vec{v}\,dR = \int_{R_L} (\vec{r} \times \vec{\omega})\,dR - \int_{B_L} \vec{r} \times (\vec{n} \times \vec{v})\,dB \qquad \text{(IV-29)}$$

where R_L is a circle of radius L bounded by B_L.

Let the flow be in the x-y plane, with the x-axis selected to be in the $\vec{\alpha}_o$ direction. On B_L, the first term of Eq. (IV-21) gives a velocity vector

$$\vec{v} = \frac{\alpha_o}{2\pi L^2} (\mathrm{Cos}\theta\, \vec{e}_r + \mathrm{Sin}\,\theta \vec{e}_\theta)$$

where $\vec{e}_r$ and $\vec{e}_\theta$ are unit vectors in the cylindrical coordinate system (r, θ) given by

$$x = r\mathrm{Cos}\theta$$

$$y = r\mathrm{Sin}\theta$$

On B_L, one has $\vec{r} = L\vec{e}_r$ and $\vec{n} = \vec{e}_r$. Thus one has

$$\vec{r} \times (\vec{n} \times \vec{v}) = -\frac{\alpha_o}{2\pi L} \mathrm{Sin}\theta\, \vec{e}_\theta$$

$$= -\frac{\alpha_o \mathrm{Sin}\theta}{2\pi L} (-\mathrm{Sin}\theta \vec{i} + \mathrm{Cos}\theta \vec{j})$$

Placing this expression into the surface integral of Eq. (IV-29), one finds this integral to be $\frac{1}{2}\vec{\alpha}_o$. Equation (IV-29) therefore gives, for two-dimensional problems,

$$\int_{R_L} \vec{v}dR = \frac{1}{2}\vec{\alpha}_o \qquad \text{(IV-30)}$$

Equation (IV-30) is independent of the size of R_L, as long as L is finite. The integral $\int_{R_\infty} \vec{v}dR$ is indeterminate if $\vec{\alpha}_o \neq 0$ because the first term on the right-hand side of Eq. (IV-20) is of order r^{-2}.

Ref. 11 gives a formula similar to Eq. (IV-30) but with the coefficient for $\vec{\alpha}_o$, $\frac{1}{2}$ in equation (IV-30), replaced by 1. That formula is derived by considering a finite cylindrical volume centered on the z-axis, bounded by two planes $z = 0$ and $z = d$, and using Eq. (IV-23). The derivation of the formula neglected the contribution of the last integral in (IV-29) (See the equation between Eqs. A-3 and A-4 of Ref. 11). The formula is in error for any finite cylinders. As it turns out, however, if the cylinder is infinite in length, then the formula in Ref. 11 is correct, as shown in Chapter V.

5. Velocity-Moment Integrals in Large Bounded Regions.

In this Section, a formula for the velocity-moment integral is derived. Using the equation

$$\vec{r} \times \vec{v} = -\frac{1}{2}r^2\vec{\omega} + \frac{1}{2}\vec{\nabla}\times(r^2\vec{v}) \qquad \text{(IV-31)}$$

one obtains, with the help of Stokes' Theorem

$$\int_R \vec{r} \times \vec{v}\, dR = -\frac{1}{2}\int_R r^2\vec{\omega}dR - \frac{1}{2}\int_B r^2(\vec{v} \times \vec{n})dB \qquad \text{(IV-32)}$$

where B is the boundary of R.

Let R be R_f', the portion of the fluid region bounded internally by the solid boundaries B_j and externally by B_L. One has from Eq. (IV-32)

$$\int_{R_f'} \vec{r} \times \vec{v} dR = -\frac{1}{2} \int_{R_f'} r^2 \vec{\omega}\, dR - \frac{1}{2} \oint_{B_L} r^2 (\vec{v} \times \vec{n}) dB - \frac{1}{2} \sum_j \oint_{B_j} r^2 (\vec{v} \times \vec{n}) dB \quad \text{(IV-33)}$$

Let R be R_j in Eq. (IV-32), one obtains

$$-\frac{1}{2} \oint_{B_j} r^2 (\vec{v} \times \vec{n}) dB = \int_{R_j} (\vec{r} \times \vec{v}) dR + \frac{1}{2} \int_{R_j} r^2 \vec{\omega}\, dR \quad \text{(IV-34)}$$

Placing Eq. (IV-34) into Eq. (IV-33) and noting that the normal vector in these two equations are directed opposite to each other, one obtains upon simplification

$$\int_{R_L} \vec{r} \times \vec{v}\, dR = -\frac{1}{2} \int_{R_L} r^2 \vec{\omega}\, dR - \frac{1}{2} \oint_{B_L} r^2 (\vec{v} \times \vec{n}) dB \quad \text{(IV-35)}$$

where R_L include the solid regions R_j and the portion of the fluid region R_f'.

Let R_L be a spherical region, or a circular region in two dimensions, centered on the origin with the radius L. Let L be sufficiently large so that the total vorticity outside R_L is negligible. One has then

$$\int_{B_L} r^2(\vec{v} \times \vec{n})dB = L^2 \oint_{B_L} \vec{v} \times \vec{n}dB = L^2 \int_{R_L} \vec{\omega}\, dR \qquad \text{(IV-36)}$$

The last integral in Eq. (IV-35) is therefore zero because of the principle of total vorticity conservation discussed in Section III-2. One obtains therefore, upon noting $r^2\,\omega$ is negligible outside R_L

$$\int_{R_L} \vec{r} \times \vec{v}\, dR = -\frac{1}{2}\vec{\beta}_o \qquad \text{(IV-37)}$$

where β_o is the integral of the second moment of vorticity defined by

$$\vec{\beta}_o = \int_{R_\infty} r_o{}^2\, \vec{\omega}_o dR_o \qquad \text{(IV-38)}$$

Equation (IV-37) relates velocity-moment integral to an integral of the second moment of vorticity. This equation is valid for both two- and three-dimensional problems.

The fact that the last integral in Eq. (IV-35) vanishes can also be shown by noting that if L is sufficiently large, then a scalar velocity potential Φ exists on B_L and is single-valued. The boundary integral in Eq. (IV-35) can therefore be expressed as

$$L^2 \oint_{B_L} \vec{\nabla}\Phi \times \vec{n}\, dB$$

Using Stokes' theorem, this integral is expressible as

$$-L^2 \int_{R_L} \vec{\nabla} \times \vec{\nabla}\Phi \, dR$$

and is zero because the integrand, being the curl of the gradient of a function, is zero.

It is not difficult to show that the terms of the asymptotic expression for $\vec{v}$, Eqs. (IV-19) and (IV-21) are all expressible as gradients of a scalar function. For example, for two-dimensional flows, the first term on the right-hand side of Eq. (IV-21) and the next term, being respectively of orders r^{-2} and r^{-3}, are important. The first term is already written in the form of a gradient of a scalar function, namely $\frac{1}{2\pi} \vec{\nabla}(\vec{\nabla} \ln\frac{1}{r} . \vec{\alpha}_o)$. The next term can be shown to be expressible as

$$\frac{1}{4\pi}\vec{\nabla} x \int_{R_L} \vec{\omega}_o r_o^2 \left[\frac{\mathrm{Cos}(2\theta_o - 2\theta)}{r^2}\right] dR_o = \frac{\vec{k}}{4\pi} x \vec{\nabla} \int_{R_L} \omega_o r_o^2 \left[\frac{\mathrm{Cos}(2\theta_o - 2\theta)}{2}\right] dR_o$$

V. AERODYNAMIC FORCE AND MOMENT

In this Chapter, the theorems and formulas presented in Chapters III and IV are utilized to derive general formulas relating the aerodynamic forces and moments to the vorticity-moment integrals.

1. Aerodynamic Force

Consider the control volume R_L bounded externally by B_L and containing the fluid occupying the region R_f' and the solid bodies occupying the regions R_j. The momentum theorem gives

$$\vec{F}_t = \frac{d}{dt}\int_{R_L} \tilde{\rho}\,\vec{v}dR + \oint_{B_L} \rho\vec{v}(\vec{v}.\vec{n})dB \qquad \text{(V-1)}$$

where $\vec{F}_t$ is the total force acting on the system within R_L, and $\tilde{\rho}$ is either the fluid density ρ or the solid density ρ_j, depending on the particular region of interest.

On account of the asymptotic behavior of $\vec{v}$, the last integral in Eq. (V-1) is negligible for sufficiently large values of L. The integral over R_L can be written as sums of integrals as follows:

$$\vec{F}_t = \rho\frac{d}{dt}\int_{R_L}\vec{v}dR - \sum_{j=1}^{N}\frac{d}{dt}\int_{R_j}\rho\vec{v}_j dR + \sum_{j=1}^{N}\frac{d}{dt}\int_{R_j}\rho_j\vec{v}_j dR \qquad \text{(V-2)}$$

The total force $\vec{F}_t$ acting on the system in R_L consists of the force $\vec{F}_L$ acting on the boundary B_L and the external force F_e:

$$F_t = \vec{F}_L + \vec{F}_e \qquad \text{(V-3)}$$

The external force $\vec{F}_e$ is the force exerted from outside the system and acts on the solid bodies. The total force acting on the solid bodies is $\vec{F}_e + \vec{F}$, where $\vec{F}$ is the aerodynamic force exerted by the fluid on the solid bodies. Newton's second law of motion gives

$$\vec{F}_e + \vec{F} = \sum_{j=1}^{N} \frac{d}{dt} \int_{R_j} \rho_j v_j dR \tag{V-4}$$

Placing Eqs. (V-3) and (V-4) into Eq. (V-2) one obtains

$$\vec{F} = \vec{F}_L - \rho \frac{d}{dt} \int_{R_L} \vec{v} dR + \sum_{j=1}^{N} \frac{d}{dt} \int_{R_j} \rho \vec{v}_j dR \tag{V-5}$$

For sufficiently large L, the shear stress is negligible on B_L and $\vec{F}_L$ is expressible in terms of p as given by Eq. (III-21). That is

$$F_L = -\rho \oint_{B_L} p\vec{n} dB = - \oint_{B_L} \left[\frac{\partial \Phi}{\partial t} - \frac{v^2}{2} + \frac{f(t)}{\rho}\right] \vec{n} dB \tag{V-6}$$

The asymptotic behavior of v shows that the term $\frac{v^2}{2}$ does not contribute to the last integral in Eq. (V-6). The function f(t), being independent of position, also does not contribute to Eq. (V-6). One thus has

$$\vec{F} = -\rho \oint_{B_L} \frac{\partial \Phi}{\partial t} \vec{n} dB - \rho \frac{d}{dt} \int_{R_L} \vec{v} dR + \rho \sum_{j=1}^{N} \frac{d}{dt} \int_{R_j} \vec{v}_j dR \tag{V-7}$$

Using Eqs. (III-20) and (IV-19), one obtains for three-dimensional flows

$$\oint_{B_L} \frac{\partial \Phi}{\partial t} \vec{n} dB = -\frac{1}{8\pi} \frac{d}{dt} \oint_{B_L} (\vec{\nabla} \frac{1}{r} \cdot \vec{\alpha}_o) \vec{n} dB$$

Expressing the last integral in a spherical coordinate system with $\vec{\alpha}_o$ directed in the $\vec{k}$ direction, one obtains

$$\oint_{B_L} \frac{\partial \Phi}{\partial t} \vec{n} dB = \frac{\vec{\alpha}_o}{8\pi} \frac{d}{dt} \int_{\theta=0}^{\pi} \int_{\Phi=0}^{2\pi} \mathrm{Cos}^2\theta \, \mathrm{Sin}\theta \, d\Phi d\theta$$

$$= \frac{\vec{\alpha}_o}{6}$$

Using Eqs. (IV-27) and the above equation, one obtains from Eq. (V-7) the following expression for aerodynamic force in three-dimensional flows.

$$\vec{F} = -\frac{1}{2}\rho \frac{d\vec{\alpha}_o}{dt} + \rho \sum_{j=1}^{N} \frac{d}{dt} \int_{R_j} \vec{v}_j dR \qquad \text{(V-8)}$$

For two-dimensional flows, using Eqs. (III-20) and (IV-21), one obtains

$$\oint_{B_L} \frac{\partial \Phi}{\partial t} \vec{n} dB = -\frac{\vec{\alpha}_o}{2\pi} \int_o^{2\pi} \mathrm{Cos}^2\theta d\theta = \frac{\vec{\alpha}_o}{2} \qquad \text{(V-9)}$$

Using Eq. (IV-30) and the above equation, one obtains from Eq. (V-7) the following expression for the aerodynamic force in two-dimensional flows:

$$\vec{F} = -\rho \frac{d\vec{\alpha}_o}{dt} + \rho \sum_{j=1}^{N} \frac{d}{dt} \int_{R_j} \vec{v}_j dR \qquad \text{(V-10)}$$

2. Moment of Aerodynamic Force

The theorems of moment of momentum gives

$$\vec{l}_t \times \vec{F}_t = \frac{d}{dt} \int_{R_L} \tilde{\rho}(\vec{r} \times \vec{v}) dR + \rho \oint_{B_L} (\vec{r} \times \vec{v})(\vec{v}.\vec{n}) dB$$

where $\vec{l}_t$ is a position vector describing the line of action of the total force $\vec{F}_t$.

The last integral in Eq. (V-11) is negligible for sufficiently large L. The total moment consists of the moment $\vec{l}_L \times \vec{F}_L$ acting on the boundary B_L and the externally applied moment $\vec{l}_e \times \vec{F}_e$. The shear stress is negligible on B_L. Consequently the moment $\vec{l}_L \times \vec{F}_L$ is negligible. One thus has

$$\vec{l}_e \times \vec{F}_e = \frac{d}{dt} \int_{R_L} \tilde{\rho}\,(\vec{r} \times \vec{v}) dR \tag{V-11}$$

or $$\vec{l}_e \times \vec{F}_e = \rho \frac{d}{dt} \int_{R_L} \vec{r} \times \vec{v} dR - \rho \frac{d}{dt} \sum_{j=1}^{N} \int_{R_j} \vec{r} \times \vec{v}_j dR$$

$$+ \frac{d}{dt} \sum_{j=1}^{N} \int_{R_j} \rho_j (\vec{r} \times \vec{v}_j) dR \tag{V-12}$$

The total moment acting on the solid bodies consists of the externally applied moment and the moment exerted by the fluid on the solid bodies. This total momentum is equal to the time rate of change of angular momentum of the solid bodies. This rate of change of angular momentum is given by the last term of Eq. (V-12). One therefore obtains the following expressions for the first moment of aerodynamic force in both two-and three-dimensional

problems, by using Eq. (IV-37)

$$\vec{1} \times \vec{F} = \frac{1}{2}\rho \frac{d\vec{\beta}_o}{dt} + \rho \sum_{j=1}^{N} \frac{d}{dt} \int_{R_j} \vec{r} \times \vec{v}_j dR \qquad \text{(V-13)}$$

where $\vec{1}$ is a position vector describing the line of action of the aerodynamic force $\vec{F}$.

Equations (V-5) and (V-10)express the aerodynamic force exerted by a fluid on solid bodies immersed in the fluid as integrals of the first moment of the vorticity vector plus an inertia term. Equation (V-13) expresses the first moment of the aerodynamic force as an integral of a second moment of the vorticity vector plus a moment of inertia term. Higher moments of the aerodynamic force can be related to integrals of higher moments of the vorticity vector.

3. Summary.

The preceeding results form a theory for aerodynamic forces and moments which encompasses much of the previous aerodynamic theories. As discussed in Chapter II of this report, this theory deals with unsteady motions of a fluid which is at rest at certain initial instant of time. A steady flow, if it exists, is considered to be approached asymptotically at large time levels. It has been shown that under this circumstance the effective vortical regions surrounding and trailing the solid bodies are of finite extent at any finite time level. The theory comprises of the following three statements:

a. The combined total vorticity of the fluid and the solid bodies is zero.

b. The aerodynamic force acting on the solid bodies is expressible as the time derivative of an integral of the first moment of the vorticity plus a inertia term.

c. The moment of aerodynamic force acting on the solid bodies is expressible in terms of the time derivative of an integral of a second moment of the vorticity plus an moment of inertia term.

These three statements are expressible mathematically as follows:

$$\int_{R_\infty} \vec{\omega}\, dR = 0 \qquad \text{(III-5), (III-}$$

where R_∞ is the infinite unlimited region jointly occupied by the fluid and all the solid bodies.

$$\vec{F} = -\frac{\rho}{d-1}\frac{d}{dt}\int_{R_\infty} \vec{r} \times \vec{\omega}\, dR + \rho \sum_{j=1}^{N} \frac{d}{dt}\int_{R_j} \vec{v}_j\, dR \qquad \text{(V-8),(V-9)}$$

where d is the dimensionality of the problem.

$$\vec{1} \times \vec{F} = \frac{\rho}{2}\frac{d}{dt}\int_{R_\infty} r^2 \vec{\omega}\, dR + \rho \sum_{j=1}^{N} \frac{d}{dt}\int_{R_j} \vec{r} \times \vec{v}_j\, dR \qquad \text{(V-13)}$$

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